Closed-Form Solutions for Nonlinear Quasi-Unsteady Transonic Aerodynamics

Gabriel A. Oyibo*
Polytechnic University, Farmingdale, New York

The existence of exact closed-form solutions for nonlinear unsteady aerodynamics is established. The full nonlinear unsteady velocity potential equations for an airfoil are considered. Evidence indicating why the traditional hodograph approach is ineffective for solving these equations is provided. Therefore, a suitable mapping scheme is employed in transforming these full nonlinear equations into the hodograph plane. A close examination of the resulting hodograph equations reveals closed-form solutions can be obtained for the nonlinear unsteady aerodynamic characteristics of an airfoil in potential flow. The shockless transonic results presented in this inviscid analysis show trends that are in agreement with the results of previous investigators and available experimental data. "Dips" were observed in the pressure distributions as the freestream Mach number is varied. It appears that there are finite optimum reduced frequencies for the pressure distributions. This result might suggest a solution to the "transonic dip" problem. Perhaps an important practical consequence of this study is the possibility of employing this approach to solve an inverse problem of designing an airfoil section with given or desired aerodynamic characteristics. Desirable candidates for such a design procedure would include supercritical oscillating shock-free or "transonic dipless" airfoil sections. Such airfoils, therefore, could be designed to meet both the performance and stability criteria simultaneously.

Nomenclature

b,c	= airfoil chord length and velocity of sound, respectively
(x,z); (u,w)	= Cartesian and hodograph coordinates, respectively
C_p,C_L	= pressure and lift coefficients, respectively
k,M	= reduced frequency and Mach number, respectively
q,P	= resultant flow velocity and pressure, respectively
J	= Jacobian of the hodograph transformation
F_m,B_m	= hypergeometric function and arbitrary constant, respectively
$Q,\bar{Q}$	= velocity potential and stream-function quantity, respectively
m,i	= integer and square root of minus one, respectively
ρ,ω	= air density and oscillation frequency, respectively
t,γ	= time and ratio of gas specific heats, respectively
ϕ,∇	= velocity potential and vector differential operator, respectively
$\chi,\bar{\psi}$	= transformed velocity potential and stream function, respectively
$()_0$	= affine space or nondimensionalized quantities
$()_\infty$	= quantities at infinity
θ	= angle inclined by velocity vector and positive x axis
$\bar{T}$	= dimensionless velocity variable

Received April 20, 1987; revision received March 19, 1988.

*Associate Professor, Aerospace Engineering Department. Member AIAA.

Introduction

THE hodograph transformation has been established as a very vital tool with which to analyze two-dimensional nonlinear steady transonic flow problems comprehensively. The significance of the hodograph representation results from the fact that whereas the linearized equations in the physical plane fail to explain certain observed transonic flow phenomena, the nonlinear equations, which more accurately describe the flow, do not seem to have simple closed-form solutions in the physical plane. With the help of suitable mapping functions, these nonlinear equations could be transformed into the hodograph plane,[1-14] where they become linear, resulting in a possibility of obtaining closed-form solutions for them. Although the hodograph approach could introduce difficulties associated with involved boundary conditions resulting from some practical problems, many investigators believe that its advantages, particularly in terms of physical insights, far outweigh its disadvantages.

Outstanding contributions from various investigators have been responsible for the development of the hodograph method for solving nonlinear potential flow problems. The early work of Molenbrock[1] and Chaplygin[2] was complemented by the later efforts of investigators like Lighthill[3] and Guderleg.[4] Consequently, researchers like Nieuwland,[5] Bauer, Garabedian, and Korn,[6] and Boersteel[7] were able to establish the hodograph approach as an effective design tool for efficient airfoils like the supercritical shockless sections. The basic idea here is to suppress the boundary-layer separation by "pushing" the shock waves on the wing toward its trailing edge and eventually diluting (or weakening) them as much as possible. Sobieczky[8,9] and his collaborators have also presented interesting results more recently.

The hodograph transformation has been known and used for over a century. Curiously, however, evidence from a literature search seems to indicate that its use has largely been restricted to the analysis of the nonlinear, steady, two-dimensional flow problems like steady transonic flow.

The reasons for this restriction or why there has not been an extension of this approach to unsteady two-dimensional or three-dimensional flow problems also appear to be absent in the literature as well. The implication, therefore, seems to be that such an approach can only be used to solve steady-state airfoil problems like performance and not dynamic instability problems (which require the unsteady flow solutions).

In transonic aeroelasticity, phenomena like the "transonic dip" require nonlinear unsteady aerodynamics for their understanding. The need to operate modern aircraft in the transonic region, among other things, has been responsible for the recent tremendous interest in transonic flow problems. Aeroelasticians and computational fluid dynamicists have been investigating these problems with a good degree of success; Refs. 15–24 are just a few examples. As a result, numerous computer codes are now available for computing transonic aerodynamics, many of them using various approximations of the full nonlinear potential equations. In spite of this progress, however, recent publications and presentations, like Ref. 24, seem to indicate that a great deal still remains to be done to thoroughly understand aeroelastic phenomena like the transonic dip. The author, inspired by a recent experience,[25] shares the view held by some investigators that a more fundamental approach could provide some of the necessary physical insights for improving our understanding of these phenomena. Such insights may not be readily extractable from an analysis in which big computer codes are used.

This paper presents the results of a preliminary investigation of the unsteady transonic aerodynamics in which the full nonlinear unsteady two-dimensional velocity potential flow equations are employed. Evidence indicating why the traditional hodograph approach is not effective for solving these equations is presented. With the help of a certain mapping scheme, the nonlinear equations are transformed into the hodograph plane. A close examination of the transformed equations reveals that if J, the Jacobian of the transformation, is prescribed, ab initio, the hodograph velocity potential satisfies a linear second-order partial differential equation. The significance of this result includes the fact that the following results appear to be possible for the first time:

1) Exact closed-form solutions for the nonlinear unsteady velocity potential can be obtained.

2) The solution of an inverse problem of designing an airfoil section with given or desired unsteady aerodynamic characteristics can be attempted.

These results, therefore, imply that it is possible to design an airfoil that can meet both the performance and dynamic stability criteria simultaneously.

The results shown in this paper are obtained by piecing the fundamental solutions in a manner similar to Nieuwland's approach.[5] From these results it appears that there are "dips" in the pressure distributions as the freestream Mach number is varied in the transonic region, a phenomenon that also has been established (Ref. 24, for example) for the aeroelastic stability characteristics. It also is observed that there appear to be finite optimum reduced frequencies for the pressure distributions. This, therefore, appears to agree with the trends established by Marble[26] for a quasi-one-dimensional flow.

Equations of Motion

By considering the concept of "control volume" for a fluid flow, it can be proved from the first principles that the physical principle of continuity of mass demands that the following equation be satisfied for any arbitrary volume:

$$\left(\frac{\partial}{\partial t}+\boldsymbol{q}\cdot\nabla\right)\rho+\rho(\nabla\cdot\boldsymbol{q})=0 \tag{1}$$

where $\boldsymbol{q}$ is the fluid-flow velocity vector, ρ is the fluid density, and ∇ is a differential vector operator.

Using a similar approach, the physical principle of the conservation of momentum can be represented by the following equation:

$$\left[\frac{\partial}{\partial t}+(\boldsymbol{q}\cdot\nabla)\right]\rho\boldsymbol{q}+\rho\boldsymbol{q}(\nabla\cdot\boldsymbol{q})=-\nabla P \tag{2}$$

By using Leibnitz's rule and Kelvin's theorem for an irrotational flow as well as the assumption that a velocity potential ϕ exists, such that

$$\boldsymbol{q}=\nabla\phi \tag{3}$$

in light of Eq. (1), Eq. (2) can be integrated to obtain the following:

$$\frac{\partial\phi}{\partial t}+\frac{\nabla\phi\cdot\nabla\phi}{2}+\int_{P_\infty}^{P}\frac{\mathrm{d}P}{\rho}=\frac{q_\infty^2}{2} \tag{4}$$

where $()_\infty$ is a reference point quantity (e.g., infinity). Equation (4) is basically the Bernoulli–Kelvin equation.

By considering isentropic flow with the following relationship between the pressure P and density

$$p\rho^{-\gamma}=\text{const} \tag{5}$$

and defining the quantity c given by

$$c^2=\frac{\mathrm{d}P}{\mathrm{d}\rho} \tag{6}$$

called the speed of sound, the Leibnitz rule can be employed to derive the following velocity potential equation from Eqs. (1) and (4):

$$a^2\nabla^2\phi-\left[\frac{\partial}{\partial t}(\nabla\phi\cdot\nabla\phi)+\frac{\partial^2\phi}{\partial t^2}+\nabla\phi-\nabla(\nabla\phi\cdot\nabla\phi)\right]=0 \tag{7}$$

When Eq. (7) is expanded and the vector calculus is carried out, the following nonlinear velocity potential equation in the physical Cartesian coordinates is obtained:

$$(c^2-u^2)\frac{\partial^2\phi}{\partial x^2}-2uw\frac{\partial^2\phi}{\partial x\,\partial z}+(c^2-w^2)\frac{\partial^2\phi}{\partial z^2}-2u\frac{\partial^2\phi}{\partial x\,\partial t}$$
$$-2w\frac{\partial^2\phi}{\partial z\,\partial t}-\frac{\partial^2\phi}{\partial t^2}=0 \tag{8}$$

where

$$u=\frac{\partial\phi}{\partial x},\qquad w=\frac{\partial\phi}{\partial z} \tag{9}$$

Once Eq. (8) is solved for ϕ, the Bernoulli–Kelvin equation (4), along with Eqs. (5) and (6), can be used to determine the pressure P as a function of the velocity potential ϕ. Therefore, in terms of ϕ, the pressure coefficient C_p, defined by

$$C_p=\frac{P-P_\infty}{\gamma P_\infty M^2} \tag{10}$$

can be expressed as

$$C_p=\frac{2}{\gamma M_\infty^2}\left[\left(1+\frac{\gamma-1}{2}M_\infty^2\right.\right.$$
$$\left.\left.\left\{1-\frac{[\nabla\phi\cdot\nabla\phi+2(\partial\phi/\partial t)]}{q_\infty^2}\right\}\right)^{\gamma/\gamma-1}-1\right] \tag{11}$$

where M_∞, the reference Mach number, is given by

$$M_\infty=\frac{q_\infty^2}{c_\infty^2} \tag{12}$$

Traditional Hodograph Methods and Exact Solutions

Early investigators like Molenbrock[1] and Chaplygin[2] were able to show that if the steady approximation of Eq. (8) were mapped onto the hodograph plane using the Legendre transformation, it is possible to determine the exact fundamental solutions (to this nonlinear equation). Of course, this was a breakthrough in the study of transonic flow. However, nothing seemed to have been said about the applicability of this type of transformation for solving the unsteady problem. In an effort to determine if the three-dimensional form of Legendre's transformation can be used for solving Eq. (8), the following results were obtained.

Consider the transformation

$$\phi = x\zeta + z\eta + t\tau - \chi \tag{13}$$

which is the Legendre transformation in three variables, where

$$\zeta = \frac{\partial\phi}{\partial x}, \qquad \eta = \frac{\partial\phi}{\partial z}, \qquad \tau = \frac{\partial\phi}{\partial t}$$

$$x = \frac{\partial\chi}{\partial\zeta}, \qquad z = \frac{\partial\chi}{\partial\eta}, \qquad t = \frac{\partial\chi}{\partial\tau} \tag{14}$$

Equation (13) transforms Eq. (8) into its counterpart in the hodograph space given by

$$\begin{aligned}(c^2-\zeta^2)[\chi_{\eta\eta}\chi_{\tau\tau}-\chi_{\eta\tau}^2]+2\zeta\eta[\chi_{\eta\zeta}\chi_{\tau\tau}-\chi_{\zeta\tau}\chi_{\eta\tau}]\\ +(c^2-\eta^2)[\chi_{\zeta\zeta}\chi_{\tau\tau}-\chi_{\zeta\tau}^2]-2\zeta[\chi_{\eta\zeta}\chi_{\tau\eta}-\chi_{\eta\eta}\chi_{\tau\zeta}]\\ -2\eta[\chi_{\zeta\zeta}\chi_{\eta\tau}-\chi_{\zeta\eta}\chi_{\zeta\eta}]-\chi_{\zeta\zeta}\chi_{\eta\eta}+\chi_{\zeta\eta}^2=0\end{aligned} \tag{15}$$

Obviously, by comparing Eqs. (8) and (15) it can be seen that Eq. (15) does not look easier than Eq. (8). This type of comparison, therefore, explains why the traditional approach is not effective for solving Eq. (8).

New Hodograph Mapping Scheme

Before formulating the new mapping scheme that can transform Eq. (8) into its relatively easier counterpart in the hodograph plane, a nondimensionalization scheme, which uses the following set of affine transformations, is considered.

Affine Transformations

$$\phi = q_\infty b\phi_0, \qquad x = bx_0, \qquad z = bz_0, \qquad t = \frac{bt_0}{q_\infty} \tag{16}$$

Equation (16) transforms Eq. (8) into its nondimensionalized counterpart in the affine space, given by

$$\begin{aligned}(c_0^2-u_0^2)\frac{\partial^2\phi_0}{\partial x_0^2}-2u_0w_0\frac{\partial^2\phi_0}{\partial x_0\partial z_0}+(c_0^2-w_0^2)\frac{\partial^2\phi_0}{\partial z_0^2}\\ -2u_0\frac{\partial^2\phi_0}{\partial x_0\partial t_0}-2w_0\frac{\partial^2\phi_0}{\partial z_0\partial t_0}-\frac{\partial^2\phi_0}{\partial t_0^2}=0\end{aligned} \tag{17}$$

Although b in Eqs. (16) can be any arbitrary length quantity, it is convenient, at least for computational purposes, to choose b as the chord length of a wing section.

Equation (17) is, therefore, the nonlinear velocity potential equation in the affine space, where

$$u_0 = \frac{\partial\phi_0}{\partial x_0}, \qquad w_0 = \frac{\partial\phi_0}{\partial z_0}, \qquad c_0 = \frac{c}{q_\infty} \tag{18a}$$

or

$$u_0 = \frac{u}{q_\infty}, \qquad w_0 = \frac{w}{q_\infty}, \qquad c_0 = \frac{c}{q_\infty} \tag{18b}$$

Hodograph Transformations

Consider ϕ_0 to be a harmonic function of the form given by

$$\begin{aligned}\phi_0(x_0,z_0,t_0) &= \Phi_0(x_0,z_0)e^{ikt_0}, \quad n\pi-\epsilon \le kt_0 \le n\pi-\epsilon\\ &= \Phi_0(x_0,z_0), \quad n\pi+\epsilon \le kt_0 \le (n+1)\pi-\epsilon\end{aligned} \tag{19}$$

where $\epsilon \to 0$, $n = 0,1,2,3,\ldots$. This is essentially a quasi-unsteady assumption. Therefore, it can be argued that this analysis is not really unsteady. Clearly a response to that argument is that the analysis is probably the first one in almost one century to show that transonic (nonlinear) flows other than two-dimensional or unsteady one-dimensional flows can be studied in the hodograph space. It thus provides an aspect of transonic flow that has never been seen before. If this assumption seems crazy to some readers, it should be noted that it is not without precedent. For example, this assumption is not likely to be crazier than the famous von Kármán–Tsien assumptions that led to the definition of a gas known as "Kármán–Tsien" gas. This gas has the ratio of the specific heats to be equal to -1, i.e., $\gamma = -1$. However, the Kármán–Tsien's approximation did provide good benchmark results that compared favorably even with experimental results. The analysis received the blessing of such important scientists as Sir M. J. Lighthill—particularly pp. 367–373 of Ref. 28.

A second example is the fact that the so-called small-disturbance theory approximation has been known to give good results. Therefore, it would seem that the primary (most important) thing in any transonic flow analysis is the *preservation* of the flow's *nonlinear character*. Every other thing seems to be secondary.

Furthermore, it should be pointed out that as a consequence of the assumptions in Eq. (19), if the steady-state solution $\phi_0(x_0,z_0)$ has shocks, such shocks would be retained in the quasi-unsteady solutions as well.

Finally, perhaps it also should be remembered that all of the transonic flow equations (and, indeed, all mathematical physics equations), including the Navier Stokes equations, are all approximations.

The nondimensionalized frequency (or Strouhal number) k is, as a result of Eqs. (16), given by

$$k = \frac{\omega b}{q_\infty} \tag{20}$$

where ω is the circular frequency of oscillations. If Eqs. (19) and (20) are substituted into Eqs. (17), the following equation is obtained:

$$\begin{aligned}(c_0^2-u_0^2)\frac{\partial^2\phi_0}{\partial x_0^2}-2u_0w_0\frac{\partial^2\phi_0}{\partial x_0\,\partial z_0}+(c_0^2-w_0^2)\frac{\partial^2\phi_0}{\partial z_0^2}+k^2\phi_0\\ =2ik(u_0^2+w_0^2)\end{aligned} \tag{21}$$

Equation (21) also can be written in the following manner:

$$\begin{aligned}(c_0^2-u_0^2)\frac{\partial^2\phi_0}{\partial x_0^2}-2u_0w_0\frac{\partial^2\phi_0}{\partial x_0\,\partial z_0}+(c_0^2-w_0^2)\frac{\partial^2\phi_0}{\partial z_0^2}+k^2\phi_0\\ =2ikq_0^2\end{aligned} \tag{22}$$

where q_0, the resultant velocity, is given by

$$q_0^2 = u_0^2 + w_0^2 \tag{23}$$

Now consider the following transformation:

$$\chi_0 = u_0x_0 + w_0z_0 - \phi_0 \tag{24}$$

where

$$x_0 = \frac{\partial\chi_0}{\partial u_0}, \qquad z_0 = \frac{\partial\chi_0}{\partial w_0} \tag{25}$$

$$\chi_0 = \bar{\chi}e^{ikt_0} \tag{26}$$

When Eqs. (24–26) are substituted into Eq. (22), the following equation in the hodograph plane must be solved to obtain the transformed velocity potential:

$$(\bar{c}^2 - \bar{u}^2)\frac{\partial^2 \bar{\chi}}{\partial \bar{w}^2} + 2\bar{u}\bar{w}\frac{\partial^2 \bar{\chi}}{\partial \bar{u}\,\partial \bar{w}} + (\bar{c}^2 - \bar{w}^2)\frac{\partial^2 \bar{\chi}}{\partial \bar{u}^2} + k^2 J\left[\bar{u}\frac{\partial \bar{\chi}}{\partial \bar{u}} + \bar{w}\frac{\partial \bar{\chi}}{\partial \bar{w}} - \bar{\chi}\right] = 0 \tag{27}$$

where

$$\bar{u} = \frac{\partial \Phi}{\partial x_0}, \qquad \bar{w} = \frac{\partial \Phi}{\partial z_0} \tag{28}$$

and $\bar{c}$ and J are the nondimensionalized steady-state velocity of sound and Jacobian of the transformation, represented by Eqs. (24–26), respectively.

From Eq. (27), it is seen that if the Jacobian J is prescribed, ab initio, the resulting *linear* equation can be solved closed-form to determine the exact fundamental solutions for the transformed potential χ and, hence, ϕ. Consequently, these solutions can be pieced together to determine the shape of an airfoil. Notice that the familiar steady hodograph equation can be recovered from Eq. (27) if k is set equal to zero. It also should be noted that the presence of the Jacobian in Eq. (27) should not necessarily be considered as an added problem in comparison to the steady hodograph equation in which the Jacobian does not appear explicitly, since its behavior must be studied in both cases in order to ensure unique solutions.

Solution Methods

Although Morawetz[25] has shown that continuous solutions for Eq. (27) (steady-state approximation was used to arrive at this conclusion) for a closed body in the transonic regime do not exist, investigators like Nieuwland[5] have shown that fundamental closed-form solutions of the steady approximation of Eq. (27) can be pieced together employing the appropriate boundary conditions to obtain interesting transonic flows over airfoils. In this paper an effort is made to obtain closed-form fundamental solutions of Eq. 27, which are consequently pieced together after the appropriate boundary conditions are enforced to study some steady and unsteady transonic flows.

Polar Coordinates

In an attempt to obtain fundamental solutions of Eq. (27), it is helpful to transform this hodograph equation of motion into a polar coordinate system.

Consider the following transformations:

$$\bar{u} = \bar{q}\cos\theta, \qquad \bar{w} = \bar{q}\sin\theta \tag{29}$$

Equations (29) transform Eq. (27) into the following polar coordinate counterpart:

$$\bar{c}^2\bar{\chi}_{\bar{q}\bar{q}} + \left[\left(\frac{\bar{c}}{\bar{q}}\right)^2 - 1\right]\bar{\chi}_{\theta\theta} + \bar{q}\left[\left(\frac{\bar{c}}{\bar{q}}\right)^2 - 1 + \left(k\frac{2\bar{J}}{\bar{q}}\right)\right]\bar{\chi}_{\bar{q}} + \left(k\frac{2\bar{J}}{\bar{q}}\right)\bar{\chi} = 0 \tag{30}$$

where $\bar{J}$ is now the Jacobian of the transformation into the polar coordinates whose relationship to J can be seen as follows:

$$J = \frac{\partial(x_0,z_0)}{\partial(\bar{u},\bar{w})} \tag{31a}$$

$$\bar{J} = \frac{\partial(x_0,z_0)}{\partial(\bar{q},0)} \tag{31b}$$

$$\frac{\partial(x,z)}{\partial(\bar{u},\bar{w})} = \frac{\partial(x_0,z_0)}{\partial(\bar{q},0)}\cdot\frac{\partial(\bar{q},0)}{\partial(\bar{u},\bar{w})} \tag{32}$$

or

$$J = \bar{J}\frac{\partial(\bar{q},0)}{\partial(\bar{u},\bar{w})} = \frac{\bar{J}}{\bar{q}} \tag{33}$$

To study the fundamental solution of Eq. (30), consider a flow in which

$$\bar{J} = J(\bar{q}) \tag{34}$$

Therefore, a general solution of the form

$$\bar{\chi} = Q(\bar{q})\cos(m\theta + \epsilon) \tag{35}$$

can be assumed for Eq. (30) in order to obtain the following equation for $Q(\bar{q})$.

$$\bar{c}^2 Q_{\bar{q}\bar{q}} + \bar{q}\left[\left(\frac{\bar{c}}{\bar{q}}\right)^2 - 1 + \left(k^2\frac{\bar{J}}{\bar{q}}\right)\right]Q_{\bar{q}} - \left\{\left[\left(\frac{\bar{c}}{\bar{q}}\right)^2 - 1\right]m^2 + \left(k^2\frac{\bar{J}}{\bar{q}}\right)\right\}Q = 0 \tag{36}$$

Equation (36) can be rewritten in a slightly different manner if the following definition is employed:

$$M = \frac{\bar{q}}{\bar{c}} \tag{37}$$

where M is the local Mach number.

Equation (36), in light of Eq. (37), becomes

$$\bar{q}^2 Q_{\bar{q}\bar{q}} + \bar{q}\left[1 - M^2 + M^2\left(k^2\frac{\bar{J}}{\bar{q}}\right)\right]Q_{\bar{q}} - \left[(1 - M^2)m^2 + M^2\left(k^2\frac{\bar{J}}{\bar{q}}\right)\right]Q = 0 \tag{38}$$

Equation (38) must be solved to obtain the transformed velocity potential $\bar{\chi}$ and, hence, the physical space velocity potential ϕ. A similar approach can be used to show that the following equation must be solved to obtain the transformed stream function given by

$$\psi = -\bar{Q}(\bar{q})\cos(m\theta + \bar{\epsilon}) \tag{39}$$

i.e.,

$$\bar{q}^2\bar{Q}_{\bar{q}\bar{q}} + \bar{q}\left[1 + M^2 + M^2\left(k^2\frac{\bar{J}}{\bar{q}}\right)\right]\bar{Q}_{\bar{q}} - \left[(1 - M^2)m^2 + M^2\left(k^2\frac{\bar{J}}{\bar{q}}\right)\right]\bar{Q} = 0 \tag{40}$$

In general, Eqs. (38) and (40) are hypergeometric and, hence, can be satisfied by combinations of power and logarithmic series. Finding solutions to the flow around a closed body, therefore, becomes dependent on piecing these types of series correctly. Employing such fundamental solutions to construct the overall solution is obviously preferable to using arbitrary series, since these fundamental solutions are "solid" in that they satisfy the equation of motion.

In order to transform Eqs. (38) and (40) into the familiar hypergeometric form, consider a reference q^* given by

$$\bar{c}^2 = \frac{\gamma + 1}{2}\bar{q}_*^2 - \frac{\gamma - 1}{2}\bar{q}^2 \tag{41}$$

and a change of independent variable given by

$$\bar{\tau} = \frac{\gamma - 1}{\gamma + 1}\left(\frac{\bar{q}}{\bar{q}^*}\right)^2 \tag{42}$$

Equation (42) transforms Eq. (40) to the following

$$\bar{\tau}(1-\bar{\tau})\bar{Q}_{\bar{\tau}\bar{\tau}} + \left[1 - \frac{(\gamma - 2 - k^2 J_0)}{\gamma - 1}\bar{\tau}\right]\bar{Q}_{\bar{\tau}} - \frac{m^2}{4\bar{\tau}}\left[1 - \frac{(\gamma + 1 - 2k^2 J_0)}{\gamma - 1}\right]\bar{Q} = 0 \tag{43}$$

An exact fundamental solution of Eq. (43) can be given by

$$\bar{Q} = B_m \bar{\tau}^{\,m/2} F_m(\bar{\tau}) \tag{44a}$$

where

$$\bar{\psi} = B_m \bar{\tau}^{m/2} F_m(\bar{\tau}) \cos(m\theta + \epsilon) \tag{44b}$$

where B_m are arbitrary constants to be determined by the boundary conditions of a particular flow problem, and $F_m(\bar{T})$ satisfy the following hypergeometric equations:

$$\bar{\tau}(1-\bar{\tau})\frac{\partial^2 F_m}{\partial \bar{\tau}^2} + \left[m + 1 - \left(m + 1 - \frac{1 + k^2 J_0}{\gamma - 1}\right)\bar{\tau}\right]\frac{\partial F_m}{\partial \bar{\tau}} + \frac{\beta}{2}(m+1)(m + k^2 J_0)F_m = 0 \tag{45}$$

where

$$\beta = \frac{1}{\gamma - 1}$$

An F_m satisfying Eq. (45) is a hypergeometric function given by

$$F_m(\bar{\tau}) = F(\bar{a},\bar{b};\bar{d};\bar{\tau}) = 1 + \frac{\bar{a}\bar{b}}{1 \cdot \bar{d}}\bar{\tau} + \frac{\bar{a}(\bar{a}+1)\bar{b}(\bar{b}+1)}{1 \cdot 2 \cdot \bar{d}(\bar{d}+1)}\bar{\tau}^2 + \cdots\cdots \tag{46}$$

where

$$\bar{a} + \bar{b} = m - \beta(1 + k^2 J_0); \qquad \bar{a}\bar{b} = -\frac{\beta}{2}(m+1)$$
$$(m + k^2 \dot{J}_0), \qquad \bar{d} = m + 1 \tag{47}$$

In terms of the velocity potential, the fundamental solution is

$$\bar{\chi} = -B_m \bar{\tau}^{\,m/2}(1-\bar{\tau})^{-\beta}\left[F_m(\bar{\tau}) + \frac{2\bar{\tau}}{m}\frac{\partial F_m(\bar{\tau})}{\partial \bar{\tau}}\right]\sin(m\theta + \bar{\epsilon}) \tag{48}$$

Equations (44) and (48) represent some exact fundamental solutions for the stream function and velocity potential, respectively, in the hodograph plane. In the examples shown in this paper, the stream function solutions were used in constructing the flow solutions.

Computed Examples

The examples computed in this paper consist of the pressure variations with Mach numbers and Strouhal numbers (reduced frequencies) for some chosen points on a flat plate in a transonic flow and what may be considered as a preliminary attempt to compute the steady and unsteady transonic pressure distributions around a 70-10-13 supercritical wing section designed in the early seventies by Bauer, Garabedian, and Korn.[6] The procedure basically consists of trying to construct compressible (transonic) flows using incompressible flows around an elliptical cylinder resulting in a shape different from the cylinder with the help of the exact fundamental potential flow solutions above. This procedure has been thoroughly documented by Nieuwland[5] for the steady flow problem. Hence, no attempt has been made in this paper to redocument the basis of this approach. Therefore, readers are referred to Ref. 5. Suffice it to say that the approach involves a lot of "bookkeeping" and patience. Furthermore, it must be pointed out that this procedure can (and should) be better automated. This endeavor is currently in progress.

The flat plate results shown in Figs. 1–6 have basically two main features:

1) Dips in the pressure distributions when the Mach numbers are varied in the transonic regime seem to agree with the general trends in the behavior of transonic characteristics such as the lift coefficient, the so-called transonic dip, which is basically the loss of aeroelastic stability at transonic speeds.

2) The pressure distribution at a particular Mach number at a particular chord length seems to have an optimum reduced frequency (Strouhal number).

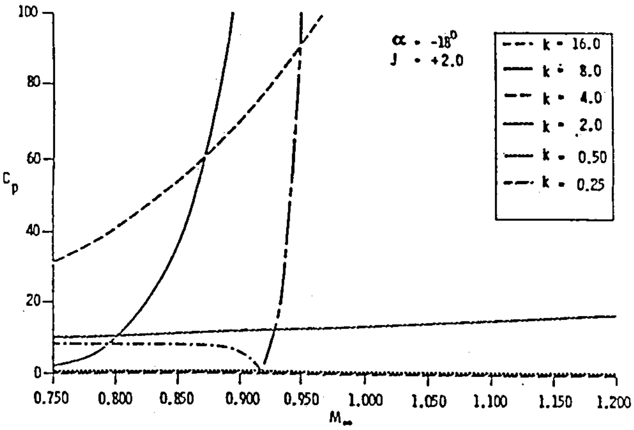

Fig. 1 Pressure coefficient vs Mach numbers for a flat plate.

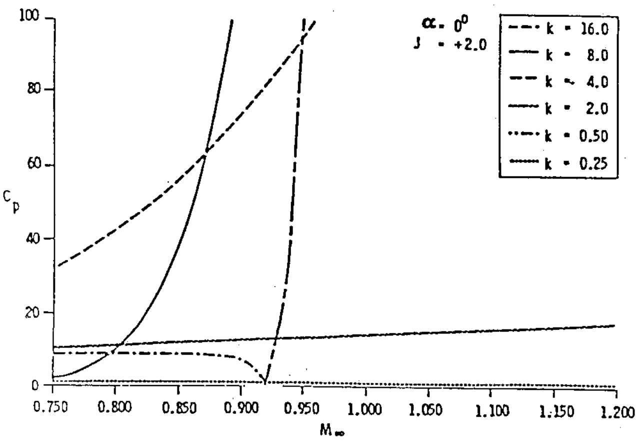

Fig. 2 Pressure coefficient vs Mach numbers for a flat plate.

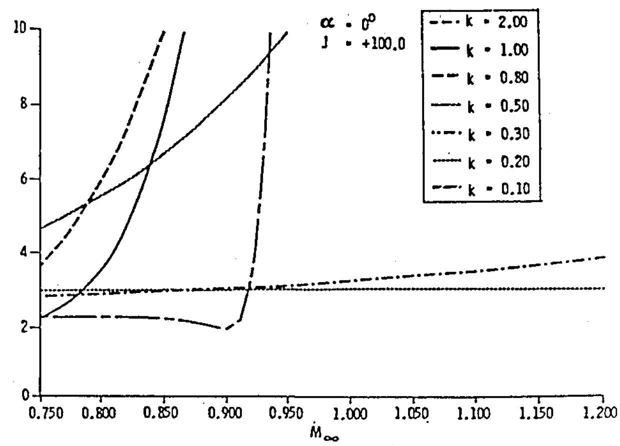

Fig. 3 Pressure coefficient vs Mach numbers for a flat plate.

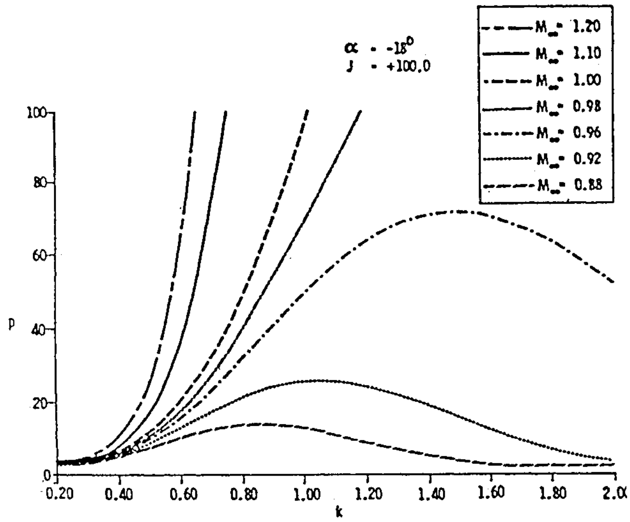

Fig. 4 Pressure coefficient vs reduced frequency for a flat plate.

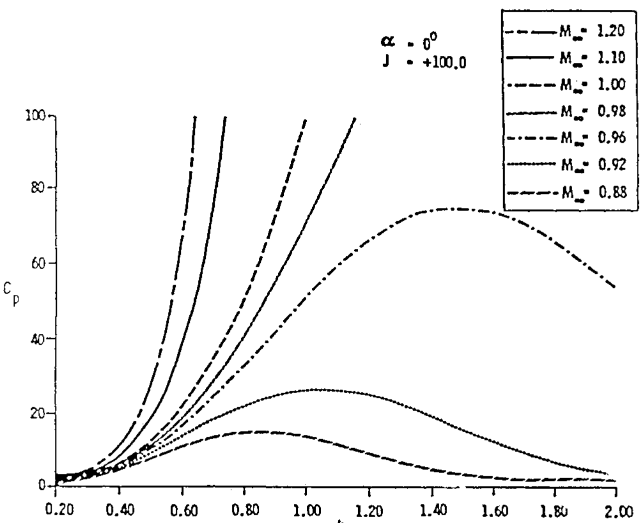

Fig. 5 Pressure coefficient vs reduced frequency for a flat plate.

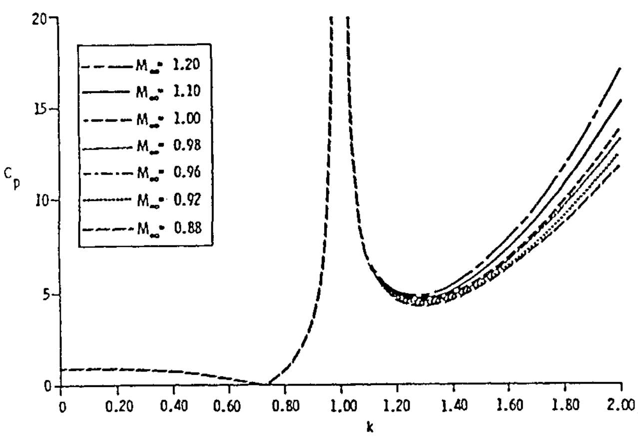

Fig. 6 Pressure coefficient vs reduced frequency for a flat plate.

Marble[26] has shown similar trends for a quasi-one-dimensional flow. The result shown in Fig. 7 for a 70-10-13 supercritical airfoil designed back in the early seventies,[6] which is preliminary in nature, seems to indicate the feasibility of using the approach outlined in this paper in designing unsteady aerodynamic characteristics for an airfoil. The pressure distribution for the supercritical airfoil, 70-10-13 shown in Fig. 7, is for a freestream Mach number of 0.7 and a lift coefficient

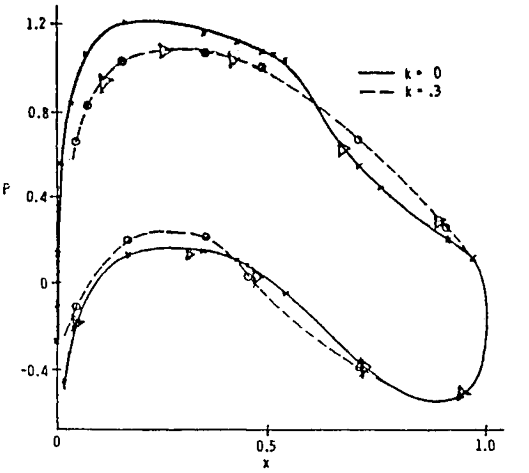

Fig. 7 Pressure distribution on a 70-10-13 supercritical airfoil.

$(C_L) = 0.98$, and the airfoil has a thickness ratio of 0.127. The angle of attack for the steady computation was 0 deg. For the unsteady computations it is assumed that $kt_0 = 0.1$, and that the airfoil is oscillating about the half-chord with a pitch amplitude of 1 deg. The main significance of Fig. 7 is that the approach outlined in this paper can be used to accurately compute the pressure distributions for supercritical airfoils, since these pressure distributions compare very well with the results obtained in Ref. 6 for the steady flow and also compare favorably with results of Ref. 15 for the unsteady flow. Because of the fact that the analysis in this paper is preliminary in nature, a more general set of conclusions would have to await more refined computations that are anticipated in the near future. Finally, it must be pointed out that the results presented in this paper are basically shockless inviscid results. Therefore, the plots shown for Mach numbers greater than unity can only be approximations to the weak shock solutions.

Concluding Remarks

This paper has attempted to establish the existence of exact closed-form fundamental solutions to the two-dimensional nonlinear unsteady potential aerodynamic equations. Evidence indicating why the traditional hodograph approach is not effective for solving the nonlinear unsteady two-dimensional flow equations is presented. Therefore, suitable mapping functions are employed to transform the nonlinear potential flow equations into the hodograph plane. An examination of the transformed flow equations in the hodograph plane reveals that if the Jacobian of the transformation is prescribed ab initio, the exact closed-form fundamental solutions for the velocity potential and stream functions can be obtained. It is seen that such Chaplygin solutions can be used in conjunction with the incompressible flow around elliptical cylinders to construct transonic flows over interesting shapes with the help of a methodology developed several decades ago by investigators such as Lighthill[3] and Nieuwland[5] for steady flows. Computations of the pressure distributions for certain points on the flat plates seem to indicate that dips exist in the pressure distributions as the freestream Mach numbers are varied in the transonic regime. The results also seem to show the existence of optimum reduced frequencies for the pressure distributions. A steady and an unsteady pressure distribution are also computed for a 70-10-13 supercritical airfoil designed back in the early seventies.[6] Although the approach needs more efficient automation, the results computed show that it is feasible to use this approach to solve the inverse problem of designing airfoils with desired unsteady aerodynamic characteristics. This, therefore, implies the possibility of designing airfoils that satisfy both the performance and stability criteria simultaneously.

Acknowledgments

This paper is dedicated to God for His inspiration. It is also dedicated to the memories of my father, S. Oyibo Obaje, and my uncle Ejima Alhaji who passed away on April 18, 1986, and May 31, 1975, respectively. The contributions of Professor P. Garabedian of the Courant Institute, New York University, New York, to this paper are acknowledged. The author also acknowledges helpful discussions with J. H. Berman and Dr. S. M. Scala, both of Fairchild Republic Company. The author is grateful to Drs. Anthony K. Amos and Arje Nochman who partially funded this work under AFOSR Contract F49620-87-C-OO46 and AFOSR Grant 89-0055.

References

[1]Molenbrock, P., "Uber einige Bewegungen eines Gases bei Annahme eines Geschwindigkeits Potentials," *Archive of Mathematical Physics*, Vol. 9, 1890, pp. 157–195.

[2]Chaplygin, S. A., "On Gas Jets," *Scientific Memorandum, Moscow University, Mathematic Physics*, No. 21, 1904, pp. 1–21; NACA TM-1063, 1944.

[3]Lighthill, M. J., "The Hodograph Transformation in Transonic Flow. II. Auxiliary Theorems on the Hypergeometric Functions. III. Flow Round a Body," *Proceedings of the Royal Society of London*, Ser. A 191, 1947, pp. 341–369.

[4]Guderley, K. G., "On Transonic Simplifications of the Hodograph Equation," Wright Air Development Center, TR-53-183, Dayton, OH, 1953.

[5]Nieuwland, G. Y., "Transonic Potential Flow around a Family of Quasi-Elliptical Aerofoil Sections," National Luchten Ruimteuaartlaboratorium Rept. TR.T172, the Netherlands, 1967.

[6]Bauer, F., Garabedian, P., and Korn, D., *Supercritical Wing Sections: Lecture Notes in Economics and Mathematical Systems*, Vol. 66, Springer-Verlag, New York, 1972.

[7]Boersteel, J. W., "A Survey of Symmetrical Transonic Potential Flows around Quasi-Elliptical Aerofoil Sections," National Luchten Ruimteuaartlaboratorium Rept. TR.T.136, the Netherlands, 1967.

[8]Sobieczky, H., "Related Analytical, Analog, and Numerical Methods in Transonic Airfoil Design," AIAA Paper 79-1556, July 1979.

[9]Hassan, A., Sobieczky, H., and Seebass, A. R., "Subsonic Airfoils with a Given Pressure Distribution," *AIAA Journal*, Vol. 22, Sept. 1984, pp. 1185–1191.

[10]Garabedian, P., *Partial Differential Equations*, Wiley, New York, 1964.

[11]Courant, R. and Hilbert, D., *Methods of Mathematical Physics*, Vol. II, Interscience, New York, 1962.

[12]Bateman, H., *Partial Differential Equations of Mathematical Physics*, Dover Publications, New York, 1944.

[13]Guderley, K., *The Theory of Transonic Flow*, Pergamon, Oxford, 1962.

[14]Garrick, I. E. and Kaplan, C., "On the Flow of a Compressible Fluid by the Hodograph Method II—Fundamental Set of Particular Flow Solutions of the Chaplygin Differential Equation," NACA Rept. No. 790, 1944.

[15]Isogai, K., "Calculation of Unsteady Transonic Flow Over Oscillating Airfoils Using the Full Potential Equation," AIAA Paper 77-448, 1977.

[16]Ballhaus, W. F. and Goorjian, P. M., "Computation of Unsteady Transonic Flows by the Indicial Method," *AIAA Journal*, Vol. 16, Feb. 1978, pp. 117–124.

[17]Yang, T. Y., Guruswamy, P., and Stiriz, A. G., "Application of Transonic Codes to Flutter Analysis of Conventional and Supercritical Airfoils," *Proceedings of the 22nd AIAA/SDM Conference*, AIAA, New York, 1981.

[18]Dowell, E. H., Bland, S. R., and Williams, M. H., "Linear/Nonlinear Behavior in Unsteady Transonic Aerodynamics," *AIAA Journal*, Vol. 21, Jan. 1983, pp. 38–46.

[19]Edwards, J. W., Bennett, R. M., Whitlow, W. Jr., and Seidel, D. A., "Time-Marching Transonic Flutter Solutions Including Angle-of-Attack Effects," *Proceedings of the 23rd AIAA/SDM Conference*, AIAA, New York, May 1982.

[20]Magnus, R. J. and Yoshihara, H., "Calculations of Transonic Flow Over an Oscillating Airfoil," AIAA Paper 75-98, Jan. 1975.

[21]Tijdeman, H. and Seebass, R., "Transonic Flow Past Oscillating Airfoils," *Annual Review of Fluid Mechanics*, Vol. 12, 1980, pp. 181–222.

[22]Whitlow, W. and Harris, W. L., "An Analytical/Numerical Method to Solve Two-Dimensional Transonic Flow," MIT Fluid Dynamics Research Lab. Rept. No. 79-3, MIT, Cambridge, MA, Aug. 1979.

[23]Dowell, E. G., Curtiss, H. C., Scanlan, R. H., and Sisto, F., *A Modern Course in Aeroelasticity*, Sijhoff and Noordhoff International Publishing, the Netherlands, 1978.

[24]Zwaan, R., "Verification of Calculation Methods for Unsteady Airloads in the Prediction of Transonic Flutter," AIAA Paper 84-087-1/2, May 1984.

[25]Oyibo, G. A., "Unified Aeroelastic Flutter Theory for Very Low Aspect Ratio Panels," *AIAA Journal*, Vol. 21, Nov. 1983, pp. 1581–1587.

[26]Marble, F. E., "Response of a Nozzle to an Entropy Disturbance—Example of Thermodynamically Unsteady Aerodynamics," *Proceedings of the Unsteady Aerodynamics Symposium*, Vol. II, Univ. of Arizona, Tucson, AZ, March 1975, pp. 699–711.

[27]Morawetz, C. S., "On the Non-Existence of Continuous Transonic Flows Past Profiles, *I. Communications of Pure and Applied Mathematics*, Vol. 9, 1956, pp. 45–68.

[28]Sears, W. R., *General Theory of High Speed Aerodynamics*, Princeton Univ. Press, Princeton, NJ, 1954.